FARTHEST POINTS IN REFLEXIVE LOCALLY UNIFORMLY ROTUND BANACH SPACES

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ABSTRACT

If S is a bounded and closed subset of a Banach space B, which is both reflexive and locally uniformly rotund, then, except on a set of first Baire category, the points in B have farthest points in S.

In the note [3] by Edelstein it is shown that, if B is a uniformly rotund Banach space and S a bounded and strongly closed subset of B, then the set

$$a(S) = \{c \in B: \exists s \in S \text{ such that } \| c - s \| \geqq \| c - x \| \forall x \in S\}$$

i.e., the set of all points in B that have farthest points in S, is dense in B. Here we will show, by a different method, that the restriction on B can be slightly relaxed and at the same time the condition on $a(S)$ strengthened. We will require B to be reflexive and locally uniformly rotund (LUR), a condition first introduced by Lovaglia [4]:

(LUR) Given $\varepsilon > 0$ and x in B, with $\| x \| = 1$, there exists a $\delta(\varepsilon, x)$ such that

$$1 - \| (x + z)/2 \| \geqq \delta$$

for all z in B such that $\| z \| \leqq 1$ and $\| x - z \| \geqq \varepsilon$.
Then $a(S)$ will be shown to contain the intersection of a denumerable family of open and dense subsets of B. By the Baire category theorem, the intersection is itself dense in B. Such a set will be called a *fat* subset of B.

We begin with a lemma on convex functions in arbitrary Banach spaces.

LEMMA 1. *If f is a convex function which satisfies a Lipschitz condition*

$$f(x) - f(y) \leqq C \| x - y \|$$

for all x and y in an arbitrary Banach space B, then there exists a fat subset E of B such that for each y in E and $\varepsilon > 0$ there exist z in B and real numbers a, k satisfying

Received July 19, 1966

$$f(x) \leqq a + k\| x - z \|^2 \text{ for all } x \text{ in } B,$$

$$f(y) > a + k\| y - z \|^2 - \varepsilon^2/k.$$

Proof. Let E_n be the set of points y in B such that

$$f(y) > a + k\| y - z \|^2 - 1/nk$$

for some triple (z, a, k) in $B \times R^2$ satisfying

$$f(x) \leqq a + k\| x - z \|^2 \text{ for all } x \text{ in } B.$$

Obviously, E_n is open. If we can show that each E_n is dense in B, then

$$E = \bigcap_{n=1}^{\infty} E_n$$

will be fat in B and it will have the required property. To see that all E_n are dense take an arbitrary z in B and a $k > 0$ and put

$$a = \sup\{f(x) - k\| x - z \|^2 : x \in B\}$$

In view of the Lipschitz condition on f it is clear that, if k is large, the only points in B that influence the supremum above are those near to z. To be specific, we have

$$a = \sup\{f(x) - k\| x - z \|^2 : x \in B \text{ and } \| x - z \| \leqq Ck^{-1}\}.$$

Choose first k sufficiently large and then y in B such that $\| y - z \| \leqq Ck^{-1}$ and such that

$$f(y) - k\| y - z \|^2 + 1/nk > a.$$

Thus E_n is dense in B and Lemma 1 is proved.

REMARK. The above lemma is a variant of the method used in [1].

Suppose now that S is an arbitrary bounded subset of B. We then define a function r on B with positive values by the formula

$$r(x) = \sup\{\| x - s \| : s \in S\}$$

Also, we will use the notation

$$B(x, a) = \{y \in B : \| y - x \| \leqq a\}.$$

for the "balls" of B.

LEMMA 2. *The function r is convex and satisfies the Lipschitz condition of Lemma 1 with $C = 1$. Moreover, for all y in B and $b \geqq 0$,*

$$\sup\{r(x) : x \in B(y, b)\} = r(y) + b.$$

Proof. r is the upper envelope of a family of convex functions, so it is convex, and it satisfies the Lipschitz condition because each of the members of the family does so, by the triangle inequality. The last statement of the lemma follows from the sup inversion formula:

$$\begin{aligned}
&\sup_x\{\sup_s\{\|x-s\|:s\in S\}:\|x-y\|\leqq b\}\\
&=\sup_s\{\sup_x\{\|x-s\|:\|x-y\|\leqq b\}:s\in S\}\\
&=\sup_s\{\|y-s\|+b:s\in S\}=r(y)+b.
\end{aligned}$$

We will now go on to investigate the differential properties of a convex function f satisfying the conditions of Lemma 1, at the points of E. A *subgradient* of f at the point y in B is an element p of B^* such that

$$f(x)\geqq f(y)+\langle p,x-y\rangle \text{ for all } x \text{ in } B.$$

The set of all subgradients of f at y is called the *subdifferential* of f at y and denoted by $\partial f(y)$. By the Hahn-Banach theorem, $\partial f(y)$ is a non-empty set for each y in B, namely, the continuity of f implies that the set $\{(x,z):f(x)<z\}$ is open in $B\times R$ and thus by the "geometric" version of the theorem (cf. Bourbaki [2], p. 69) separated by a hyperplane from the point $(y,f(y))$.

LEMMA 3. *Let f satisfy the conditions of Lemma 1. Then if y is in E, the elements of $\partial f(y)$ all have the same norm in B^*.*

COROLLARY. *Take $f=r$ and y in E. Then the elements of $\partial r(y)$ all have norm one.*

Proof. We may suppose that $\partial f(y)$ contains some elements $p\neq 0$, and we choose $\varepsilon\leqq\|p\|/4$. Combining the inequalities, we get

$$\langle p,x-y\rangle<\varepsilon^2/k+k(\|x-z\|^2-\|y-z\|^2) \text{ for all } x \text{ in } B. \tag{1}$$

Take the sup of each side of (1). For a given value of $\|x-y\|$, we get the most out of the right hand side of (1) by putting $x-y=\lambda(y-z)$ for some $\lambda\geqq 0$, then

$$\lambda\|p\|\;\|y-z\|\leqq\varepsilon^2/k+k(2\lambda+\lambda^2)\|y-z\|^2.$$

Now put $\lambda=\varepsilon/k\|y-z\|$, the result is

$$\|p\|\leqq 2k\|y-z\|+2\varepsilon. \tag{2}$$

If, instead, we take the inf of both sides of (1) and want to get the smallest value to the right of (1) for a fixed $\|x-y\|$, we put $y-x=\lambda(y-z)$ and $0\leqq\lambda\leqq 1$, because

$$\|x-z\|\geqq\|y-z\|-\|y-x\|$$

with equality in the above case. Thus

$$-\lambda\|p\|\,\|y-z\| \leqq \varepsilon^2/k + k(-2\lambda+\lambda^2)\|y-z\|^2.$$

Again put $\lambda = \varepsilon/k\|y-z\|$, this is possible since by (2) and the condition $\varepsilon \leqq \|p\|/4$ we have $\varepsilon \leqq k\|y-z\|$. We get an inequality which combined with (2) gives

$$(3) \qquad \big|\, \|p\| - 2k\|y-z\| \,\big| \leqq 2\varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this proves Lemma 3.

To see that the Corollary follows from Lemma 3, we use the estimate of Lemma 2 instead of the subgradient relation in (1) and proceed as before, the result is

$$(2') \qquad 1 \leqq 2k\|y-z\| + 2\varepsilon$$

Combined with (3), this shows that $\|p\| \geqq 1$ whereas $C = 1$ in the Lipschitz condition for r proves the opposite inequality.

We can now prove the statement announced in the beginning.

THEOREM. *If the reflexive Banach space B is* (LUR) *and S is a bounded and closed subset of B, then $a(S)$ is fat in B.*

Proof. We will in fact prove that the set $a(S)$ contains the earlier constructed set E for the function r on B. Suppose, then, that y is in E and take p in $\partial r(y)$. The functional p assumes its minimum (since B is reflexive) on $B(y, r(y))$ at a point x. We will show that x is in S and hence a farthest point to y in S. With no loss of generality we assume that $y = 0$ and $r(y) = 1$, i.e., $\|x\| = 1$.

Consider now the function r on the segment from 0 to $-x$. By the corollary to Lemma 3 the increase of r on this segment is exactly one. Hence the ball $B(-x, 2)$ is the smallest ball with center $-x$ that contains S, so we may find points z in S very near to the boundary of this ball, i.e.

$$1 - \|(x+z)/2\|$$

can be made arbitrary small. Say that it is made smaller than the quantity $\delta(\varepsilon, x)$ in the condition (LUR); it then follows that $\|x - z\| < \varepsilon$ because $\|z\| \leqq 1$. But ε can be chosen arbitrarily small, hence x is the strong limit of points z in S, so by hypothesis x is in S, as claimed.

REFERENCES

1. E. Asplund, *Fréchet differentiability of convex functions*, To appear.

2. N. Bourbaki, *Espaces vectoriels topologiques*, Chap. I–II, Paris 1953.

3. M. Edelstein, *Farthest points of sets in uniformly convex Banach spaces*. Israel J. of Math. Vol. 4, (1966), 171-176.

4. A. R. Lovaglia, *Locally uniformly convex Banach spaces*, Trans. Amer. Math. Soc. **78**, (1955), 225-238.

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